



ELSEVIER

Linear Algebra and its Applications 310 (2000) 139–148

**LINEAR ALGEBRA
AND ITS
APPLICATIONS**

www.elsevier.com/locate/laa

Commuting pairs and triples of matrices and related varieties

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Received 22 June 1999; accepted 10 February 2000

Submitted by C.-K. Li

Abstract

In this note, we show that the set of all commuting d -tuples of commuting $n \times n$ matrices that are contained in an n -dimensional commutative algebra is a closed set, and therefore, Gerstenhaber's theorem on commuting pairs of matrices is a consequence of the irreducibility of the variety of commuting pairs. We show that the variety of commuting triples of 4×4 matrices is irreducible. We also study the variety of n -dimensional commutative subalgebras of $M_n(F)$, and show that it is irreducible of dimension $n^2 - n$ for $n \leq 4$, but reducible, of dimension greater than $n^2 - n$ for $n \geq 7$. © 2000 Published by Elsevier Science Inc. All rights reserved.

Keywords: Commuting matrices; Commuting variety; Commutative subalgebras of matrix algebras

1. Introduction

Let F be an algebraically closed field. We are interested in the affine variety of commuting d -tuples of elements in the ring $M_n(F)$ of $n \times n$ matrices over F . Denote this variety by $\mathcal{C}(d, n, F) = \mathcal{C}(d, n)$.

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¹ Partly supported by an N.S.F. grant.

The first major result about this variety was due to Motzkin and Taussky [6] in a fundamental paper. They proved that:

Theorem 1. $\mathcal{C}(2, n)$ is irreducible.

Gerstenhaber [2] reproved this result and obtained other results. In particular, the following result was proved in [2].

Theorem 2. If A, B are commuting elements of $M_n(F)$, then A and B are contained in an n -dimensional commutative subalgebra of $M_n(F)$.

See [7] for a nice proof of the result. Note two immediate corollaries of this:

Theorem 3. Let A and B be commuting elements of $M_n(F)$ and let \mathcal{A} be the algebra generated by A and B .

1. $\dim \mathcal{A} \leq n$; and
2. $\dim \text{Cent}(\mathcal{A}) \geq n$.

There were several proofs of Part (1) of Theorem 3 above given in [1,5,10]. It was pointed out in [3] that Theorem 3 follows very easily from Theorem 1 (both these conditions are closed conditions on $\mathcal{C}(2, n)$ and they hold on the nonempty open subset, where A has distinct eigenvalues — by Theorem 1, this subset is dense, whence the results).

In this note, we set up a certain incidence correspondence in a suitable product of projective varieties, and then project onto one component to find that the pairs of commuting matrices that are contained in some commutative subalgebra of $M_n(F)$ of dimension n is a closed subset of $\mathcal{C}(2, n)$. Since the nonempty open subset of commuting pairs, where the first matrix has distinct eigenvalues, is contained in this closed subset, all of $\mathcal{C}(2, n)$ must be in this set. We thus find that Theorem 2 is also a consequence of Theorem 1.

Projecting from the correspondence onto a different component, we find that the set of commutative subalgebras of $M_n(F)$ of dimension n has the natural structure of an algebraic variety: it is a closed set in the Grassmanian variety of n -dimensional subspaces of n^2 dimensional space. We study this variety and show that for $n = 2, 3$, and 4 , it is an irreducible variety of dimension $n^2 - n$, and that for general n , it has a component of dimension $n^2 - n$. We show that for $n \geq 7$, this variety is not irreducible.

We also study $\mathcal{C}(3, n)$. Gerstenhaber [2] had proved that $\mathcal{C}(d, n)$ is reducible for $d \geq 4$ and $n \geq 4$ and had asked if $\mathcal{C}(3, n)$ is irreducible. The first author had proved [3] that $\mathcal{C}(3, n)$ is reducible for $n \geq 32$, but irreducible for $n \leq 3$. Using results from [8], we show that $\mathcal{C}(3, 4)$ is also irreducible.

2. The incidence correspondence

We use elementary algebraic geometry to study the incidence relation between d -tuples of commuting matrices and commutative subalgebras of $M_n(F)$ of dimension n .

Write $\mathcal{C}_{\mathbf{P}}(d, n)$ for the projectivization of the affine variety of commuting d -tuples of $n \times n$ matrices. (Note that the commutativity of matrices is a homogenous condition on the entries of the matrices.) We will denote the equivalence class of a commuting d -tuple A_1, \dots, A_d by $(A_1 : \dots : A_d)$. Recall that the set of n -dimensional subspaces of an n^2 -dimensional space has the natural structure of a projective variety via the Plucker embedding which associates to any n -dimensional space W with basis v_1, \dots, v_n , the point $[W] = v_1 \wedge \dots \wedge v_n$ in $\mathbf{P}(\bigwedge^n(V))$ (see [4] for instance; this embedding is independent of the basis v_1, \dots, v_n chosen). Denote this variety by $\mathbf{G}(n, n^2)$. In the product variety $\mathcal{C}_{\mathbf{P}}(d, n) \times \mathbf{G}(n, n^2)$, consider the set X of all points $((A_1 : \dots : A_d), [W])$ subject to the following conditions:

1. Each A_i is in the subspace W .
2. W is closed under multiplication.
3. The multiplication in W is commutative.
4. The identity matrix I_n is in W .

Each of these is a polynomial condition on $\mathcal{C}_{\mathbf{P}}(d, n) \times \mathbf{G}(n, n^2)$. For instance, the first and fourth conditions simply read $A_i \wedge W = 0$ and $I_n \wedge W = 0$ in $\bigwedge^{n+1}V$, and of course, these wedge products can be read off the coordinates of A_i and the coordinates of the point $[W]$. As for the second and third conditions, note that there exists an open cover $\{U_i\}$ of $\mathbf{G}(n, n^2)$ such that on each U_i , a basis for W can be read off the coordinates of the point $[W]$. (This can be gleaned from the discussion in [4, p. 65]. For instance, a point $[W]$ for which the first coordinate is nonzero can be represented as the row space of a unique matrix of the form $(I_n|M)$, where M is an $n \times (n^2 - n)$ matrix. Note that the entries of M differ by just a sign from suitable $n \times n$ subdeterminants of the matrix $(I_n|M)$. Thus, the coordinates of a basis for W differ by just a sign from suitable Plucker coordinates of $[W]$.) If b_k ($k = 1, \dots, n$) are the basis vectors represented in terms of the coordinates of a point $[W]$ in some open set of this cover, then the second condition translates on this open set to $(b_k b_l) \wedge W = 0$ in $\mathbf{P}(\bigwedge^{n+1}(V))$ for all k and all l , while the third condition translates on this open set to $b_k b_l = b_l b_k$ for all k and all l . It follows that X is locally closed in $\mathcal{C}_{\mathbf{P}}(d, n) \times \mathbf{G}(n, n^2)$, and hence closed [9, Chapter 1, Section 4.2].

Since X is a closed projective variety, the projections $\pi_1 : X \rightarrow \mathcal{C}_{\mathbf{P}}(d, n)$ and $\pi_2 : X \rightarrow \mathbf{G}(n, n^2)$ have closed images [9, Chapter 1, Section 5.2]. We thus have the following:

Proposition 4.

1. The set of commutative d -tuples that are contained in some n -dimensional commutative subalgebra of $M_n(F)$ of dimension n is a closed subset of $\mathcal{C}(d, n)$.

2. The set of commutative subalgebras of $M_n(F)$ of dimension n is a closed subset of $\mathbf{G}(n, n^2)$.

Proof. 1. The set of commutative d -tuples that are contained in some n -dimensional commutative subalgebra of $M_n(F)$ of dimension n is just the cone over $\pi_1(X)$.

2. This was essentially proved above (see the discussions on Conditions 2–4). Note that the set of commutative subalgebras of $M_n(F)$ of dimension n is precisely $\pi_2(X)$. \square

Remark 5. It is clear that similar considerations apply to subalgebras of $M_n(F)$ of any dimension d , satisfying any set of polynomial identities.

3. Commuting d -tuples of matrices

Let us write $\mathcal{Z}(d, n)$ for the set of commutative d -tuples that are contained in some n -dimensional commutative subalgebra of $M_n(F)$ of dimension n . It now follows very easily from Proposition 4 that Theorem 2 is a consequence of Theorem 1: the nonempty open subset of $\mathcal{C}(2, n)$, where the first matrix has distinct eigenvalues is contained in $\mathcal{Z}(2, n)$; since $\mathcal{C}(2, n)$ is irreducible, this set is dense in $\mathcal{C}(2, n)$, and therefore, since $\mathcal{Z}(2, n)$ is closed, all of $\mathcal{C}(2, n)$ must be contained in $\mathcal{Z}(d, n)$.

More generally, let $X_1(d, n)$ be the nonempty open subset of $\mathcal{C}(d, n)$, where the first matrix is nonderogatory. (Recall that a matrix A in $M_n(F)$ is nonderogatory if $[F[A] : F] = n$, equivalently, if $\text{Cent}(A) = F[A]$. This condition, of course, includes matrices with distinct eigenvalues.) Let $X(d, n)$ denote the closure of $X_1(d, n)$. Let $Y(d, n)$ denote the subset of $\mathcal{C}(d, n)$ where the last $d - 1$ matrices commute with a nonderogatory matrix. The proof of Theorem 1, as found, for instance, in [3], consists of showing that $X_1(2, n)$ is irreducible, and thus that $X(2, n)$ is irreducible, and then showing that $Y(2, n)$ is contained in $X(2, n)$. Since every matrix commutes with a nonderogatory matrix, we find $Y(2, n) = \mathcal{C}(2, n)$, and thus, $\mathcal{C}(2, n) = X(2, n)$ is irreducible.

The same proof techniques apply to show the following.

Proposition 6. $X_1(d, n)$, and hence $X(d, n)$, is irreducible of dimension $n^2 + (d - 1)n$, and $Y(d, n) \subseteq X(d, n)$.

Proof. If (A_1, \dots, A_d) is in $X_1(d, n)$, then there exist uniquely determined polynomials f_2, \dots, f_d of degree at most $n - 1$ such that $A_i = f_i(A_1)$ ($i = 2, \dots, d$). Thus, $X_1(d, n)$ is parametrized by the nonderogatory matrices and $d - 1$ copies of the space of polynomials of degree at most $n - 1$, from which the irreducibility and dimension follow. Also, if A_2, \dots, A_d commute with a nonderogatory matrix B , then the line $((1 - t)A_1 + tB, A_2, \dots, A_d)$ is contained in $\mathcal{C}(d, n)$, and a nonempty subset of the line is contained in the open set $X_1(d, n)$. Hence, the entire line must be in $X(d, n)$, and in particular, so must the point (A_1, \dots, A_d) . \square

The following is a consequence of Proposition 4 above.

Theorem 7. *Let A_1, \dots, A_d be in $X(d, n)$. Let $\mathcal{A} = F[A_1, \dots, A_d]$. Then*

1. \mathcal{A} is contained in an n -dimensional commutative subalgebra of $M_n(F)$.
2. The dimension of \mathcal{A} is at most n , while the dimension of the centralizer of \mathcal{A} is at least n .

Proof. If A_1 is nonderogatory, then $\mathcal{A} = F[A_1]$. It follows that $X_1(d, n)$ is contained in $\mathcal{Z}(d, n)$, so the closure $X(d, n)$ is also contained in $\mathcal{Z}(d, n)$. Statement 1 now follows. Statement 2 is an immediate consequence of Statement 1. \square

We now turn our attention to $\mathcal{C}(3, 4)$, the variety of commuting triples of 4×4 matrices. As explained in Section 1, it was shown by the first author [3] that $\mathcal{C}(3, n)$ is reducible for $n \geq 32$ but irreducible for $n \leq 3$. Using a result from [8], we prove the following:

Theorem 8. *$\mathcal{C}(3, 4)$ is an irreducible variety.*

Proof. As proved in [8, Theorem 15], the set of all commuting triples (A, B, C) in which some two of A, B , and C commute with a 2-regular matrix is contained in the closure of $X_1(3, n)$. (Recall that a matrix is i -regular if each eigenspace is at most i -dimensional.) Since this closure (denoted $X(3, n)$ above and \overline{U}_1 in [8]) is irreducible, it is sufficient to show that any two commuting 4×4 matrices commute with a 2-regular matrix to be able to conclude that $\mathcal{C}(3, 4)$ is irreducible.

Let A and B be two commuting 4×4 matrices. We may assume that $V \cong F^4$ is an indecomposable $F[A, B]$ module, since otherwise, the problem reduces to showing that any two commuting matrices of size at most 3 commute with a 2-regular matrix, and here, the problem is trivial. Thus, A and B have only one eigenvalue, and subtracting this off, we may assume that A and B are both nilpotent. Moreover, we may assume that A and B are not 2-regular, and that neither is a scalar. Thus, both A and B are nilpotent matrices that are 3-regular but not 2-regular, so the Jordan form for each is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, each has a three-dimensional nullspace and a one-dimensional image contained in the nullspace. The common kernel of A and B , by a dimension count, must be at least two-dimensional.

It follows very easily from the Jordan form of the matrices that any one-dimensional subspace of V that is invariant under A must be contained in the kernel of A (and similarly for B as well). Since BV is one-dimensional and $A(BV) = B(AV) \subset$

BV , we find that BV is contained in the kernel of A . Thus, $A(BV) = B(AV) = 0$, and of course, $A(AV) = B(BV) = 0$, so both AV and BV are one-dimensional subspaces contained in the common kernel of A and B .

Let W be a two-dimensional subspace of V contained in the common kernel of A and B that contains both AV and BV . Extend a basis w_1, w_2 of W to a basis w_1, w_2, w_3, w_4 of V . In this basis, consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(the i th column represents $M(w_i)$). This is a 2-regular matrix, and $AM = MA = 0$, and $BM = MB = 0$. This proves the theorem. \square

We point out an immediate corollary.

Corollary 9. *If A, B , and C are any three commuting 4×4 matrices, the algebra $F[A, B, C]$ is contained in some commutative subalgebra of $M_4(F)$ of dimension 4. Hence, $F[A, B, C]$ has dimension at most 4, and the centralizer of $F[A, B, C]$ has dimension at least 4.*

Proof. With the irreducibility of $C(3, 4)$ at hand, this is now standard: the closure of $X_1(3, 4)$ must be all of $C(3, 4)$, and Theorem 7 now applies. \square

Remark 10. The irreducibility of $C(3, n)$ is still open for $5 \leq n \leq 31$.

4. The variety of n -dimensional commutative subalgebras of $M_n(F)$

By Proposition 4, the set of commutative subalgebras of $M_n(F)$ of dimension n has the structure of a closed set in $\mathbf{G}(n, n^2)$. Let A_n denote this variety.

Our first result is a lower bound for the dimension of A_n .

Proposition 11. *For all $n \geq 2$, the closure in A_n of the set of algebras generated by nonderogatory matrices is an irreducible variety of dimension $n^2 - n$. In particular, the dimension of A_n is at least $n^2 - n$.*

Proof. Let $U \subseteq \mathbf{A}^{n^2}$ be the open set of nonderogatory matrices. We define a map $\phi: U \rightarrow A_n$ by sending the matrix A to the algebra generated by A ; explicitly, ϕ sends A to $1 \wedge A \wedge \cdots \wedge A^{n-1}$. The fiber over $F[A]$ is the open set of nonderogatory matrices in the algebra $F[A]$, and is thus of dimension n . Since U is irreducible of dimension n^2 , the closure of $\phi(U)$ must be irreducible of dimension $n^2 - n$. \square

For small values of n , we can sharpen Proposition 11 further.

Theorem 12. *For $n \leq 4$, A_n is an irreducible variety of dimension $n^2 - n$. In particular, for $n \leq 4$, the algebras generated by nonderogatory matrices are dense in A_n .*

Proof. By [3] and Theorem 8 above, the set of commuting triples of $n \times n$ matrices is an irreducible variety for $n = 2, 3$, and 4. For $n \leq 4$, let $U \subseteq C(3, n)$ be the open set of triples (A, B, C) such that the F -algebra generated by A , B , and C has dimension exactly n . For each subset T of $\{(i, j, k) \mid 0 \leq i, j, k \leq n-1\}$ of cardinality n , let $U_T \subseteq U$ be the open set of triples (A, B, C) such that the matrices $\{A^i B^j C^k \mid (i, j, k) \in T\}$ are linearly independent. (So, U is simply the union over all T of the U_T .) We define $\phi_T: U_T \rightarrow A_n$ by sending the triple (A, B, C) to $\bigwedge_{(i,j,k) \in T} A^i B^j C^k$. The morphisms ϕ_T on U_T patch together to give a morphism $\phi: U \rightarrow A_n$ that sends a triple (A, B, C) to the algebra $F[A, B, C]$. Since $n \leq 4$, every algebra in A_n needs at most three generators, so ϕ is surjective. For any point $F[A, B, C]$ in A_n , the fiber over the point is the intersection of $F[A, B, C] \times F[A, B, C] \times F[A, B, C]$ with U , an open set of dimension $3n$. Since U is irreducible of dimension $n^2 + 2n$ (this follows from the irreducibility of $C(3, n)$ for $n \leq 4$ and Proposition 6), we find that $\phi(U) = A_n$ is irreducible of dimension $n^2 - n$. The last statement of the theorem follows from this and Proposition 11. \square

Before we investigate A_n for higher n , we need an observation.

Lemma 13. *For any positive integer d , the set of algebras in A_n that can be generated by d elements is an open set of A_n .*

Proof. As described in the discussions before Proposition 4, $\mathbf{G}(n, n^2)$ has an open cover $\{U_i\}$ such that in each U_i , there is an explicit formula for determining a set of basis vectors of a subspace W from the coordinates of the point $[W] \in U$. Fix one such open set U , and let b_j ($j = 1, \dots, n$) be a basis of W represented as a function of the coordinates of $[W]$. Let $x_{i,j}$ ($1 \leq i \leq d$, $1 \leq j \leq n$) be a new set of indeterminates, and let $B_i = \sum_j x_{i,j} b_j$ ($1 \leq i \leq d$) be d generic matrices in W . Form the $n^2 \times n^d$ matrix whose columns are the entries of $B_1^{i_1} \cdots B_d^{i_d}$ ($0 \leq i_1, \dots, i_d \leq n-1$). The various $n \times n$ subdeterminants of this matrix are polynomials in the variables $x_{i,j}$, and the condition that there be some set of generators of W of cardinality d is precisely the condition that not all these polynomials be identically zero, that is, that not all coefficients of all these polynomials vanish. This is an open set condition on U , and the union of such open sets over all the U_i is precisely the set of d generated algebras. \square

Corollary 14. *For each $n \geq 2$, A_n has a component of dimension $n^2 - n$. In particular, if A_n is irreducible, it must have dimension $n^2 - n$.*

Proof. Lemma 13 above shows that the one-generated algebras form an open set in A_n , and Proposition 11 shows that this open set must be irreducible. Hence, this set must be contained wholly within some component of A_n , and being open in that component, must be dense. It follows that that component must have dimension $n^2 - n$. The last statement is clear. \square

Theorem 15. *For $n \geq 7$, A_n is not irreducible.*

Proof. Write $n = 2k$ for n even, and $n = 2k + 1$ for n odd. Write V for F^n , and let v_1, \dots, v_n be a basis for V . Let X be the variety $\{[W] \in \mathbf{G}(n-1, n^2) \mid f \cdot g = 0 \forall f, g \in W \text{ and } \dim(F\text{-span}\{\text{image}(f) \mid f \in W\}) \leq k\}$. (Here, we identify F^{n^2} with $\text{End}_F(V)$, so the product $f \cdot g$ is a product of matrices. One shows that X is a variety by writing out a basis for W in terms of its Plucker coordinates — see the discussions preceeding Proposition 4 — and observing that both conditions on $[W]$ are polynomial conditions.) Let U be the open set of this variety consisting of all $[W]$ that satisfy the additional requirement that $\dim(F\text{-span}\{\text{image}(f) \mid f \in W\})$ is exactly k . For each $[W] \in U$, let $S_W = F\text{-span}\{\text{image}(f) \mid f \in W\}$. We have a map $\phi: U \rightarrow \mathbf{G}(k, n)$ given by $[W] \mapsto [S_W]$. We claim that this map is a surjective morphism.

On an open set V of $\mathbf{G}(n-1, n^2)$, let f_1, \dots, f_{n-1} be a basis for a subspace W belonging to U , written in terms of the Plucker coordinates of $[W]$. For each subset T of $\{(i, j) \mid 1 \leq i \leq n-1, 1 \leq j \leq n\}$ of cardinality k , let V_T be the open set of V where the vectors $f_i(v_j)$, $(i, j) \in T$, are linearly independent. Then $\phi([W]) = \bigwedge_{(i,j) \in T} f_i(v_j)$ on V_T , so ϕ is a morphism. Now given any subspace Y of V of dimension k , note that the set of $f \in \text{End}_F(V)$ such that $f(V) \subseteq Y$, and $f(Y) = 0$ can be written in a suitable basis as block 2×2 matrices, with the $(1, 1)$, $(2, 1)$ and $(2, 2)$ blocks being zero. (The product of any two of these matrices is zero.) It is now trivial to find a subspace of dimension $n-1$ of these matrices the span of whose images is all of Y . The surjectivity of ϕ is now clear.

The fiber over any $[Y]$ in $\mathbf{G}(k, n)$ is the set $\{[W] \in \mathbf{G}(n-1, n^2) \mid f(V) \subseteq Y, f(Y) = 0 \forall f \in W\}$ and $F\text{-span } f(V) = Y, f \in W\}$. By writing matrices in a suitable basis as in the paragraph above, one sees that this is an open set of $\mathbf{G}(n-1, k^2)$ when n is even, and of $\mathbf{G}(n-1, k(k+1))$ when n is odd. If U' is an irreducible component of U such that $\phi(U')$ is dense in $\mathbf{G}(k, n)$ (such a component must exist, as $\mathbf{G}(k, n)$ is irreducible), we find by [9, Chapter 1, Section 6.3, Theorem 7] that $\dim(U')$ is the sum of $\dim(\mathbf{G}(k, n))$ and the dimension of the fibers. Thus, in the even case, we find $\dim(U) \geq \dim(U') = k^2 + (2k-1)(k-1)^2$, and in the odd case, $\dim(U) \geq \dim(U') = k(k+1) + 2k(k^2-k)$.

But we have a map $U \rightarrow A_n$ that sends $[W]$ to the algebra $W + F \cdot I_n$. It is easy to see that this map is a morphism, and using the fact that any $f \in W$ is nilpotent, it is easy to see that the map is injective. Hence, $\dim(A_n) \geq \dim(U)$. For $n \geq 8$ in the even case, and $n \geq 7$ in the odd case, we find that $\dim(A_n)$ is greater than $n^2 - n$. By Corollary 14, A_n cannot be irreducible for $n \geq 7$. \square

Example 16. For $n = 2$, we can determine the equations of A_2 quite explicitly.

Let \mathcal{A} be a two-dimensional subalgebra of $M_2(F)$ (such an algebra is necessarily commutative). Let

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$M = \begin{pmatrix} x & a \\ b & y \end{pmatrix}$$

be a basis. Then \mathcal{A} is represented by the 2×2 subdeterminants of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ x & a & b & y \end{pmatrix}$$

in $\mathbf{G}(2, 4) \subseteq \mathbf{P}^5$. Thus, A is represented by the point $(a : b : y - x : 0 : -a : -b)$. Letting z_i ($i = 1, \dots, 6$) stand for the i th homogenous coordinate on \mathbf{P}^5 , the subspace corresponding to \mathcal{A} lives in the linear subspace of \mathbf{P}^5 given by $z_1 + z_5 = 0$, $z_2 + z_6 = 0$, and $z_4 = 0$. Conversely, given a point in this linear subspace of \mathbf{P}^5 , note that not all of z_1, z_2 and z_3 can be zero, so I_2 and any matrix of the form

$$M = \begin{pmatrix} u & z_1 \\ z_2 & z_3 + u \end{pmatrix}$$

(for arbitrary $u \in F$) are linearly independent and hence generate a two-dimensional algebra. The point corresponding to this algebra is precisely $(z_1 : z_2 : z_3 : 0 : -z_1 : -z_2)$.

We thus find: The set of two-dimensional subalgebras of $M_2(F)$ corresponds to the two-dimensional linear subspace of \mathbf{P}^5 given by the equations $z_1 + z_5 = 0$, $z_2 + z_6 = 0$, and $z_4 = 0$.

Since a matrix of the form

$$\begin{pmatrix} u & z_1 \\ z_2 & z_3 + u \end{pmatrix}$$

(for arbitrary $u \in F$) has distinct eigenvalues iff $z_3^2 + 4z_1z_2 \neq 0$, we also find: The open subset of this linear subspace given by $z_3^2 + 4z_1z_2 \neq 0$ corresponds to algebras generated by matrices with distinct eigenvalues, while the subvariety $z_3^2 + 4z_1z_2 = 0$ corresponds to algebras generated by conjugates of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One can show numerically that A_3 and A_4 are not linear subvarieties of $\mathbf{P}(\bigwedge^3(F^3))$ and $\mathbf{P}(\bigwedge^4(F^4))$ by showing that the linear span of the set of algebras generated by nonderogatory matrices has dimension higher than (respectively) $3^2 - 3$ and $4^2 - 4$.

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